Isomorphism classes of A-hypergeometric systems

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Abstract

For a finite set A of integral vectors, Gel'fand, Kapranov and Zelevinskii defined a system of differential equations with a parameter vector as a D-module, which system is called an *A-hypergeometric* (or a *GKZ hypergeometric*) *system*. Classifying the parameters according to the D-isomorphism classes of their corresponding A-hypergeometric systems is one of the most fundamental problems in the theory. In this paper we give a combinatorial answer for the problem, and illustrate it in two particularly simple cases: the normal case and the monomial curve case.

1 Introduction

For a finite set A of integral vectors, Gel'fand, Kapranov and Zelevinskii defined a system of differential equations with a parameter vector as a D-module, which system is called an *A-hypergeometric* (or a *GKZ hypergeometric*) *system* ([5]). Many authors studied D-invariants of the A-hypergeometric systems: In Cohen-Macaulay case, Gel'fand, Kapranov and Zelevinskii determined the characteristic cycles ([6]) and proved the irreducibility of the monodromy representations for nonresonant parameters ([4]); Adolphson proved the rank of an A-hypergeometric system equals the volume of the convex hull of A in the semi-nonresonant case ([1]); The author, Sturmfels and Takayama scrutinized the ranks in [13]; Cattani, D'Andrea, and Dickenstein determined rational solutions and algebraic solutions in monomial curve case ([2]), and recently Cattani, Dickenstein, and Sturmfels in [3] considered when an A-hypergeometric system has a rational solution other than Laurent polynomial solutions.

The purpose of this paper is to classify A-hypergeometric systems with respect to D-isomorphisms. This is one of the most fundamental problems in the theory. Under the assumption that the finite set A lies in a hyperplane off the origin, we shall give a combinatorial answer for this problem, and illustrate it in two particularly simple cases: the normal case and the monomial curve case.

Throughout the paper, we consider the finite set A fixed. In Section 2, we define a finite set $E_\tau(\beta)$ for a parameter β and a face τ of the cone generated by A. Then our main theorem (Theorem 2.1) states that two A-hypergeometric

systems corresponding to parameters β and β' are D-isomorphic if and only if $E_\tau(\beta)$ equals $E_\tau(\beta')$ for all faces τ. In Section 2, we prove the only-if-part of the theorem and state some basic properties of the set $E_\tau(\beta)$.

Sections 3 and 4 are devoted to the study of the algebra of contiguity operators, which algebra is called the *symmetry algebra*. In Section 3, we summarize some known facts on the symmetry algebra. We introduce the *b-ideals* in Section 4 and prove their elements correspond to contiguity operators. Furthermore we describe each b-ideal in terms of the standard pairs of a certain monomial ideal. Using this description, we give the proof of the if-part of our main theorem in the end of Section 4.

In Sections 5 and 6, we illustrate our main theorem in the normal case and the monomial curve case respectively, since the theorem reduces to relatively simple forms in both cases.

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2 Main theorem

We work over a field $\mathbf{k}$ of characteristic zero. Let $A = (a_1, \ldots, a_n) = (a_{ij})$ be an integer $d \times n$-matrix of rank d. We assume that all a_j belong to one hyperplane off the origin in $\mathbf{Q}^d$. We denote by I_A the toric ideal in $\mathbf{k}[\partial] = \mathbf{k}[\partial_1, \ldots, \partial_n]$, that is

$$I_A = \langle \partial^u - \partial^v \mid Au = Av,\, u, v \in \mathbf{N}^n \rangle \subset \mathbf{k}[\partial].$$

For a column vector $\beta = {}^t(\beta_1, \ldots, \beta_d) \in \mathbf{k}^d$, let $H_A(\beta)$ denote the left ideal of the Weyl algebra

$$D = \mathbf{k}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

generated by I_A and $\sum_{j=1}^n a_{ij}\theta_j - \beta_i$ $(i = 1, \ldots, d)$ where $\theta_j = x_j \partial_j$. The quotient $M_A(\beta) = D/H_A(\beta)$ is called the *A-hypergeometric system with parameter β*.

We denote the set $\{ a_1, \ldots, a_n \}$ by A as well. Let τ be a face of the cone

$$\mathbf{Q}_{\geq 0} A = \{ \sum_{j=1}^n c_j a_j \mid c_j \in \mathbf{Q}_{\geq 0} \}. \tag{2.1}$$

For a parameter $\beta \in \mathbf{k}^d$, we consider the following set:

$$E_\tau(\beta) := \{ \lambda \in \mathbf{k}(A \cap \tau)/\mathbf{Z}(A \cap \tau) \mid \beta - \lambda \in \mathbf{N}A + \mathbf{Z}(A \cap \tau) \}. \tag{2.2}$$

Here $\mathbf{N} = \{ 0, 1, 2, \ldots \}$ and we agree that $\mathbf{k}(A \cap \tau) = \mathbf{Z}(A \cap \tau) = \{ 0 \}$ when $\tau = \{ 0 \}$.

The following is the main theorem in this paper.

Theorem 2.1 *The A-hypergeometric systems $M_A(\beta)$ and $M_A(\beta')$ are isomorphic as D-modules if and only if $E_\tau(\beta) = E_\tau(\beta')$ for all faces τ of the cone $\mathbf{Q}_{\geq 0} A$.*

Before the proof, we recall the formal series solutions ϕ_v defined in [13]. For $v \in \mathbf{k}^n$, its *negative support* $\mathrm{nsupp}(v)$ is the set of indices i with $v_i \in \mathbf{Z}_{<0}$. When $\mathrm{nsupp}(v)$ is minimal with respect to inclusions among $\mathrm{nsupp}(v+u)$ with $u \in \mathbf{Z}^n$ and $Au = 0$, v is said to have *minimal negative support*. For v with minimal negative support, we define a formal series

$$\phi_v = \sum_{u \in N_v} \frac{[v]_{u_-}}{[v+u]_{u_+}} x^{v+u}. \tag{2.3}$$

Here

$$N_v = \{\, u \in \mathbf{Z}^n \mid Au = 0,\ \mathrm{nsupp}(v) = \mathrm{nsupp}(v+u)\,\},$$

and $u_+, u_- \in \mathbf{N}^n$ satisfy $u = u_+ - u_-$ with disjoint supports, and $[v]_w = \prod_{j=1}^n v_j(v_j-1)\cdots(v_j-w_j+1)$ for $w \in \mathbf{N}^n$. Proposition 3.4.13 of [13] states that the series ϕ_v is a formal solution of $M_A(Av)$.

Proof. Here we prove the only-if-part of the theorem. The proof of the if-part will be given in the end of Section 4.

We suppose that $\lambda \in E_\tau(\beta) \setminus E_\tau(\beta')$, and we shall prove $M_A(\beta)$ and $M_A(\beta')$ are not isomorphic.

Represent λ as $\sum_{a_j \in \tau} l_j a_j$. Consider the direct product

$$R_{\tau,\lambda} := \prod_{u \in \mathbf{Z}^n,\, u_j \in \mathbf{N}\ (a_j \notin \tau)} \mathbf{k} x^{l+u}.$$

Here we put $l_j = 0$ for $a_j \notin \tau$. Note that $R_{\tau,\lambda}$ has the natural D-module structure. There exists $u \in \mathbf{Z}^n$ with $u_j \in \mathbf{N}$ $(a_j \notin \tau)$ such that $\beta = A(l+u)$ and $l+u$ has minimal negative support. Then the series $\phi_{l+u} \in R_{\tau,\lambda}$ is a formal solution of $M_A(\beta)$, and hence $\mathrm{Hom}_D(M_A(\beta), R_{\tau,\lambda}) \neq 0$. On the other hand, $\mathrm{Hom}_D(M_A(\beta'), R_{\tau,\lambda}) = 0$ since $A(l+u) \neq \beta'$ for any $u \in \mathbf{Z}^n$ with $u_j \in \mathbf{N}$ $(a_j \notin \tau)$. Therefore $M_A(\beta)$ and $M_A(\beta')$ are not isomorphic. $\square$

In the remainder of this section, we collect some properties of the set $E_\tau(\beta)$. We call a face of $\mathbf{Q}_{\geq 0}A$ of dimension $d-1$, a facet. Recall that for a facet σ the linear form F_σ satisfying the following conditions is unique and called the *primitive integral support function*:

1. $F_\sigma(\mathbf{Z}A) = \mathbf{Z}$,
2. $F_\sigma(a_j) \geq 0$ for all $j = 1, \ldots, n$,
3. $F_\sigma(a_j) = 0$ for all $a_j \in \sigma$.

Proposition 2.2 *1. Each $E_{\mathbf{Q}_{\geq 0}A}(\beta)$ consists of one element. The equality $E_{\mathbf{Q}_{\geq 0}A}(\beta) = E_{\mathbf{Q}_{\geq 0}A}(\beta')$ means $\beta - \beta' \in \mathbf{Z}A$.*

2. $E_{\{0\}}(\beta) = \{0\}$ or $\emptyset$. $E_{\{0\}}(\beta) = \{0\}$ if and only if $\beta \in \mathbf{N}A$.

3. For a facet σ, $E_\sigma(\beta) \neq \emptyset$ if and only if $F_\sigma(\beta) \in F_\sigma(\mathbf{N}A)$.

4. *For faces $\tau \subset \sigma$, there exists a natural map from $E_\tau(\beta)$ to $E_\sigma(\beta)$. In particular, if $E_\tau(\beta) \neq \emptyset$, then $E_\sigma(\beta) \neq \emptyset$.*

5. *For any $\chi \in \mathbf{N}A$, there exists a natural inclusion from $E_\tau(\beta)$ to $E_\tau(\beta+\chi)$.*

Proof. All statements follow directly from the definition of $E_\tau(\beta)$. $\Box$

Proposition 2.3 *1.*

$$|E_\tau(\beta)| \leq [(\mathbf{Q}(A \cap \tau)) \cap \mathbf{Z}A : \mathbf{Z}(A \cap \tau)]. \tag{2.4}$$

2. *Assume $(\mathbf{Q}(A\cap\tau))\cap\mathbf{Z}A = \mathbf{Z}(A\cap\tau)$. If $\beta-\beta' \in \mathbf{Z}A$, and if neither $E_\tau(\beta)$ nor $E_\tau(\beta')$ is empty, then $E_\tau(\beta) = E_\tau(\beta')$.*

Proof.

1. Let $\lambda, \lambda' \in E_\tau(\beta)$. Then $\lambda - \lambda' \in (\mathbf{k}(A \cap \tau)) \cap \mathbf{Z}A$. By Cramér's formula, $(\mathbf{k}(A \cap \tau)) \cap \mathbf{Z}A = (\mathbf{Q}(A \cap \tau)) \cap \mathbf{Z}A$.

2. Let $E_\tau(\beta) = \{ \lambda \}$, $E_\tau(\beta') = \{ \lambda' \}$. Since $\beta - \beta' \in \mathbf{Z}A$, there exist $\chi, \chi' \in \mathbf{N}A$ such that $\beta+\chi = \beta'+\chi'$. Then $\{ \lambda \} = E_\tau(\beta+\chi) = E_\tau(\beta'+\chi') = \{ \lambda' \}$ by Proposition 2.2 (5).

$\Box$

Example 2.4 *Let*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

There are four facets:

$$\begin{aligned} \sigma_{12} : &= \mathbf{Q}_{\geq 0}a_1 + \mathbf{Q}_{\geq 0}a_2, & (2.5)\\ \sigma_{23} : &= \mathbf{Q}_{\geq 0}a_2 + \mathbf{Q}_{\geq 0}a_3, & (2.6)\\ \sigma_{34} : &= \mathbf{Q}_{\geq 0}a_3 + \mathbf{Q}_{\geq 0}a_4, & (2.7)\\ \sigma_{14} : &= \mathbf{Q}_{\geq 0}a_1 + \mathbf{Q}_{\geq 0}a_4, & (2.8) \end{aligned}$$

and four one-dimensional faces: $\mathbf{Q}_{\geq 0}a_1, \ldots, \mathbf{Q}_{\geq 0}a_4$. For all faces τ but σ_{14}, the indices $[(\mathbf{Q}(A\cap\tau))\cap\mathbf{Z}A : \mathbf{Z}(A\cap\tau)]$ are one. Hence for $\beta \in \mathbf{N}A$, $E_\tau(\beta) = \{ 0 \}$ for all faces $\tau \neq \sigma_{14}$. The quotient $(\mathbf{Q}(A \cap \sigma_{14})) \cap \mathbf{Z}A/\mathbf{Z}(A \cap \sigma_{14})$ has two elements and can be represented by 0 and ${}^t(1,1,0)$. Since $a_2 - {}^t(1,1,0) = a_3 - a_4$, and $a_3 - {}^t(1,1,0) = a_2 - a_1$, we obtain $E_{\sigma_{14}}(a_2) = E_{\sigma_{14}}(a_3) = \{ 0, {}^t(1,1,0) \}$. Proposition 2.2 (5) implies that for $\beta \in \mathbf{N}A$, $E_{\sigma_{14}}(\beta) = \{ 0 \}$ if and only if $\beta \in \mathbf{N}a_1 + \mathbf{N}a_4$, otherwise $E_{\sigma_{14}}(\beta) = \{ 0, {}^t(1,1,0) \}$. Therefore $\mathbf{N}A$ splits into two isomorphism classes in this case.

Recall that a parameter β is said to be *nonresonant* (respectively *semi-nonresonant*) if $\beta \notin \mathbf{Z}A + \mathbf{k}(A \cap \sigma)$ (respectively $\beta \notin (\mathbf{Z}A \cap \mathbf{Q}_{\geq 0}A) + \mathbf{k}(A \cap \sigma)$) for any facet σ, or equivalently, if $F_\sigma(\beta) \notin \mathbf{Z}$ (respectively $F_\sigma(\beta) \notin \mathbf{N}$) for any facet σ. Hence the nonresonance implies the semi-nonresonance.

Proposition 2.5 *If β is semi-nonresonant, then $E_\tau(\beta) = \emptyset$ for all proper faces τ of $\mathbf{Q}_{\geq 0}A$.*

Proof. The semi-nonresonace clearly implies $E_\sigma(\beta) = \emptyset$ for all facets σ. Proposition 2.2 (4) finishes the proof. $\Box$

Corollary 2.6 *Let β and β' be semi-nonresonant. Then $M_A(\beta)$ and $M_A(\beta')$ are isomorphic if and only if $\beta - \beta' \in \mathbf{Z}A$.*

Proposition 2.7 *If a parameter β satisfies*

$$E_\tau(\beta) = \emptyset \quad \text{for all proper faces } \tau, \tag{2.9}$$

then

1. *for any $\chi \in \mathbf{N}A$, $M_A(\beta - \chi)$ is isomorphic to $M_A(\beta)$.*
2. *Recall that all elements of A lie on one hyperplane H off the origin. We normalize the volume of a polytope on H so that a simplex whose vertices affinely span the lattice $H \cap \mathbf{Z}A$ has volume one. Then the rank of $M_A(\beta)$, i.e. the rank of the solution sheaf of $M_A(\beta)$, equals the volume of the convex hull of A.*

Proof. (1) By Proposition 2.2 (5), $E_\tau(\beta - \chi) = \emptyset$ for all proper faces τ. Hence by Proposition 2.2 (1), we deduce the statement from Theorem 2.1.

The proof of (2) is the same as that of Theorem 4.5.2 of [13] (p. 185). $\Box$

3 Symmetry algebra

We consider the algebra of contiguity operators, which algebra is called the symmetry algebra. It controls isomorphisms among A-hypergeometric systems with different parameters. We have investigated the symmetry algebra of normal A-hypergeometric systems in [11]. The proofs of some results in [11] remain valid without the normality condition. In this section, we summarize such results.

Let

$$\tilde{S} := \{\, P \in D \mid I_A P \subset D I_A \,\}.$$

Then $\tilde{S}$ is an associative algebra and $\tilde{S} \cap DI_A$ is its two-sided ideal. We call $S := \tilde{S}/\tilde{S} \cap DI_A$ the *symmetry algebra* of A-hypergeometric systems. The symmetry algebra S is nothing but the algebra $\mathrm{End}_D(D/DI_A)$. We remark that D/DI_A can be considered as the system of differential equations for the generating functions of A-hypergeometric functions.

In what follows, we denote simply by P, the element of D/DI_A represented by $P \in D$. For $\chi \in \mathbf{N}A$, all ∂^u with $Au = \chi$ represent the same element in D/DI_A. Hence we sometimes denote it by ∂^χ.

Proposition 3.1 *1. $\partial_1, \ldots, \partial_n \in S$.*

2. $\sum_{j=1}^n a_{ij}\theta_j \in S$ for all $i = 1, \ldots, d$.

3. The morphism from the polynomial ring $\mathbf{k}[\,s\,] = \mathbf{k}[\,s_1, \ldots, s_d\,]$ to S mapping s_i to $\sum_{j=1}^n a_{ij}\theta_j$ $(i = 1, \ldots, d)$ is injective.

Proof. See Lemma 1.1 in [11] for (1) and (2), and Corollary 1.3 in [11] for (3). ▯

We consider that $\mathbf{Z}A$ is the character group of the algebraic torus $T = \{\,(t_1, \ldots, t_d) \,|\, t_1, \ldots, t_d \in \mathbf{k}^\times\,\}$. Let N be the dual group of $\mathbf{Z}A$, and $s_1, \ldots, s_d$ the basis of $\mathbf{k} \otimes_{\mathbf{Z}} N$ dual to the standard basis of $\mathbf{k}^d = \mathbf{k} \otimes_{\mathbf{Z}} \mathbf{Z}A$. Under the identification of $\mathbf{k} \otimes_{\mathbf{Z}} N$ with the Lie algebra of T ([8]), each s_i equals $t_i \frac{\partial}{\partial t_i}$. The morphism in Proposition 3.1 (3) is induced from the differential of the injective morphism:

$$T \ni t \longmapsto (t^{a_1}, \ldots, t^{a_n}) \in (\mathbf{k}^\times)^n. \tag{3.10}$$

We thus consider $\mathbf{k}[\,s\,]$ as a subspace of S and, accordingly, as a subspace of D/DI_A. For each $\chi \in \mathbf{Z}A$, we define the weight space S_χ with weight χ by

$$S_\chi := \{\, P \in S \,|\, [s, P] = \chi(s)P \quad (\forall s \in N)\,\}.$$

Here the bracket $[P, Q]$ means $PQ - QP$.

Remark 3.2 *Note that the multiplication by $P \in S_\chi$ from the right defines a D-homomorphism from $M_A(\beta + \chi)$ to $M_A(\beta)$. Hence $P(\psi_\beta)$ is a solution of $M_A(\beta + \chi)$ for a solution ψ_β of $M_A(\beta)$. In this sense, the operator P is a contiguity operator shifting parameters by χ.*

Theorem 3.3 *1. The symmetry algebra S has no zero-divisors.*

2. The symmetry algebra S has the following weight space decomposition:

$$S = \bigoplus_{\chi \in \mathbf{Z}A} S_\chi. \tag{3.11}$$

3. The weight space S_0 equals the polynomial ring $\mathbf{k}[\,s\,]$.

4. For each $\chi \in \mathbf{N}A$, the weight space $S_{-\chi}$ equals $\mathbf{k}[\,s\,]\partial^\chi$.

Proof. See Lemma 1.4, and Propositions 2.3, 2.4, 2.9 in [11]. ▯

The following proposition will be used in the next section.

Proposition 3.4 (Proposition 2.6 in [11]) *The natural morphism*

$$D/DI_A \longrightarrow \mathbf{k}\langle\, x, \partial^{\pm}\,\rangle/\mathbf{k}\langle\, x, \partial^{\pm}\,\rangle I_A$$

is injective where $\mathbf{k}\langle\, x, \partial^{\pm}\,\rangle$ *is the algebra generated by* D *and* $\partial_1^{-1}, \ldots, \partial_n^{-1}$ *with relations* $[\, x_i, \partial_j^{-1}\,] = \delta_{ij}\partial_j^{-2}$ $(i, j = 1, \ldots, n)$.

4 b-Ideals

We have seen in Theorem 3.3 that the symmetry algebra S has a weight decomposition with respect to $\mathbf{Z}A$, and that each S_χ for $-\chi \in \mathbf{N}A$ is the free $\mathbf{k}[s]$-module of rank one with basis $\partial^{-\chi}$. Next we wish to compute the weight space S_χ for arbitrary χ. Suppose that $E \in S_\chi$ and $\chi = \chi_+ - \chi_-$ with $\chi_+, \chi_- \in \mathbf{N}A$. Then the operator $E\partial^{\chi_+}$ belongs to $S_{-\chi_-}$. Hence by Theorem 3.3 (4), there exists a polynomial $b \in \mathbf{k}[s]$ such that $E\partial^{\chi_+} = b\partial^{\chi_-}$. Such polynomials b varying $E \in S_\chi$ form an ideal of $\mathbf{k}[s]$. We shall define the *b-ideal* B_χ below to be such an ideal.

Fix any $\chi \in \mathbf{Z}A$, and define an ideal I_χ of $\mathbf{k}[\partial]$ by

$$I_\chi := I_A + M_\chi \tag{4.12}$$

where

$$M_\chi := \langle\, \partial^u \mid Au \in \chi + \mathbf{N}A\,\rangle. \tag{4.13}$$

Define the ideal B_χ of *b-polynomials* by

$$B_\chi := \mathbf{k}[s] \cap DI_\chi. \tag{4.14}$$

Proposition 4.1 *Let* $\chi = \chi_+ - \chi_-$ *with* $\chi_+, \chi_- \in \mathbf{N}A$. *For* $b \in B_\chi$, *there exists a unique operator* $E \in S_\chi$ *such that* $b\partial^{\chi_-} = E\partial^{\chi_+}$. *The operator* E *is independent of the expression* $\chi = \chi_+ - \chi_-$.

Moreover any operator in S_χ *can be obtained in this way.*

Proof. Since $b\partial^{\chi_-} \in DI_\chi\partial^{\chi_-} \subset DI_A + D\partial^{\chi_+}$, there exists an operator $E \in D$ such that $b\partial^{\chi_-} = E\partial^{\chi_+}$. The uniqueness, the independence, and $E \in S_\chi$ follow from the equality $E = b\partial^{\chi}$ in $\mathbf{k}\langle x, \partial^{\pm}\rangle$ and Proposition 3.4.

Let $E \in S_\chi$ and $\chi = \chi_+ - \chi_-$ with $\chi_+, \chi_- \in \mathbf{N}A$. Then $E\partial^{\chi_+} \in S_{-\chi_-}$. By Theorem 3.3 (4), there exists a polynomial $b \in \mathbf{k}[s]$ such that $E\partial^{\chi_+} = b\partial^{\chi_-}$. Then $b \in I_\chi$ and thus $b \in B_\chi$. ∎

We have the following algorithm of obtaining the operator $E \in S_\chi$ corresponding to $b \in B_\chi$, which generalizes Algorithm 3.4 in [12].

Algorithm 4.2 *Let $\chi = Au - Av$ and $u, v \in \mathbf{N}^n$.*
Input: a polynomial $b \in B_\chi$.
Output: an operator $E \in S_\chi$ with $E\partial^u = b\partial^v$.

1. *For $i = 1, \ldots, n$, compute a Gröbner basis $\mathcal{G}_i$ of I_A with respect to any reverse lexicographic term order with lowest variable ∂_i.*
2. *Expand $b(\sum_j a_{1j}\theta_j, \ldots, \sum_j a_{dj}\theta_j)\partial^v$ in $\mathbf{Q}\langle x, \partial\rangle$ into a $\mathbf{Q}$-linear combination of monomials $x^l\partial^m$.*
3. *$i := 1$, $E :=$ the output of Step 2.*
   While $i \leq n$, do
   (a) *Reduce E modulo $\mathcal{G}_i$ in $\mathbf{Q}\langle x, \partial\rangle$.*
   (b) *The output of Step 3-(a) has $\partial_i^{u_i}$ as a right factor. Divide it by $\partial_i^{u_i}$.*
   (c) *$i := i + 1$, $E :=$ the output of Step 3-(b).*

The proof of the correctness is completely analogous to that of Algorithm 3.4 in [12].

We thus reduce the study of S_χ to that of B_χ, and for the study of $B_\chi = \mathbf{k}[s] \cap DI_\chi$, we study $\mathbf{k}[\theta] \cap DI_\chi$ first. Since M_χ is the largest monomial ideal in I_χ, we have by Lemma 4.4.4 in [13],

Proposition 4.3
$$\mathbf{k}[\theta] \cap DI_\chi = \widetilde{M_\chi} \tag{4.15}$$
where $\widetilde{M_\chi}$ is the distraction of M_χ, i.e., $\widetilde{M_\chi} = \mathbf{k}[\theta] \cap DM_\chi$.

For the study of $\widetilde{M_\chi}$, we recall the standard pairs of a monomial ideal. Let M be a monomial ideal of $\mathbf{k}[\partial]$. Then a pair (u, τ) with $u \in \mathbf{N}^n$ and $\tau \subset \{1, \ldots, n\}$ is called a *standard pair* of M if it satisfies the following conditions:

1. $u_j = 0$ for all $j \in \tau$. (We abbreviate this to $u \in \mathbf{N}^{\tau^c}$, where ^{c} stands for taking the complement.)
2. There exists no $v \in \mathbf{N}^\tau$ such that $\partial^{u+v} \in M$.
3. For each $j \notin \tau$, there exists $v \in \mathbf{N}^{\tau \cup \{j\}}$ such that $\partial^{u+v} \in M$.

For an algorithm of obtaining the set of standard pairs, see [7]. Let $\mathcal{S}(M_\chi)$ denote the set of standard pairs of M_χ. By Corollary 3.2.3 in [13], the distraction $\widetilde{M_\chi}$ is described as follows:

$$\widetilde{M_\chi} = \bigcap_{(u,\tau) \in \mathcal{S}(M_\chi)} \langle\, \theta_i - u_i \,|\, i \notin \tau \,\rangle. \tag{4.16}$$

Lemma 4.4 *Let (u,τ) be a standard pair of M_χ. Then $A\mathbf{Q}^\tau_{\geq 0} := \sum_{j\in\tau}\mathbf{Q}_{\geq 0}a_j$ is a proper face of $\mathbf{Q}_{\geq 0}A$, and moreover $\tau = \{\, i \,|\, a_i \in A\mathbf{Q}^\tau_{\geq 0}\,\}$.*

Proof. Suppose that $A\mathbf{Q}^\tau_{\geq 0}$ is not contained in any facet of $\mathbf{Q}_{\geq 0}A$. Then there exists $\gamma \in A\mathbf{N}^\tau := \sum_{j\in\tau}\mathbf{N}a_j$ such that $F_\sigma(\gamma) > 0$ for all facets σ. Then $F_\sigma(Au + m\gamma) \gg 0$ for $m \gg 0$ and all facets σ. By Lemma 1 in the appendix of [14], $Au + m\gamma \in \chi + \mathbf{N}A$ for $m \gg 0$. This contradicts (u,τ) is a standard pair of M_χ.

Next we claim $(A\mathbf{Q}^{\tau^c}_{\geq 0}) \cap (A\mathbf{Q}^\tau) = \{\,0\,\}$, which implies the lemma. Suppose $(A\mathbf{Q}^{\tau^c}_{\geq 0}) \cap (A\mathbf{Q}^\tau) \neq \{\,0\,\}$. Let $v \in \mathbf{N}^{\tau^c}$ be a nonzero element satisfying $Av \in A\mathbf{Z}^\tau$. Then there exists $w \in \mathbf{N}^\tau$ such that $Aw \in Av + A\mathbf{N}^\tau$. Since $A(u+mw) \notin M_\chi$ for any $m \in \mathbf{N}$, $(Au + A\mathbf{N}^{\tau\cup\tau'}) \cap M_\chi = \emptyset$ for $\tau' = \{i \,|\, v_i \neq 0\}$. This contradicts (u,τ) is a standard pair of M_χ again. ∎

Thanks to Lemma 4.4, we regard the set τ of a standard pair (u,τ) as the proper face $A\mathbf{Q}^\tau$ of $\mathbf{Q}_{\geq 0}A$.

For an ideal I of $\mathbf{k}[s]$, we denote by $V(I)$ the zero set of I. Proposition 4.3 and the equation (4.16) give the following prime decomposition of B_χ and irreducible decomposition of the zero set $V(B_\chi)$.

Theorem 4.5 *1.*

$$B_\chi = \bigcap_{(u,\tau)\in\mathcal{S}(M_\chi)} \langle\, F_\sigma - F_\sigma(Au) \,|\, \sigma \text{ :facet} \supset \tau\rangle. \tag{4.17}$$

2.

$$V(B_\chi) = \bigcup_{(u,\tau)\in\mathcal{S}(M_\chi)} (Au + \mathbf{k}(A\cap\tau)). \tag{4.18}$$

Proof. From (4.16), we only need to show

$$\mathbf{k}[s] \cap \langle \theta_i - u_i \,|\, i \notin \tau\rangle = \langle F_\sigma - F_\sigma(Au) \,|\, \sigma \supset \tau\rangle. \tag{4.19}$$

First we have

$$V(\mathbf{k}[s] \cap \langle \theta_i - u_i \,|\, i \notin \tau\rangle) = Au + \mathbf{k}(A\cap\tau) = V(\langle F_\sigma - F_\sigma(Au) \,|\, \sigma \supset \tau\rangle). \tag{4.20}$$

Hence

$$\begin{aligned} \mathbf{k}[s] \cap \langle \theta_i - u_i \,|\, i \notin \tau\rangle &\supset \langle F_\sigma - F_\sigma(Au) \,|\, \sigma \supset \tau\rangle \\ &= I(V(\langle F_\sigma - F_\sigma(Au) \,|\, \sigma \supset \tau\rangle)) \\ &= I(V(\mathbf{k}[s] \cap \langle \theta_i - u_i \,|\, i \notin \tau\rangle)) \end{aligned} \tag{4.21}$$

where I stands for taking the defining ideal. On the other hand, $J \subset I(V(J))$ is automatic for any ideal J. We therefore obtain (4.19). ∎

Proposition 4.6 *1.*

$$V(B_{\chi+\chi'}) \subset V(B_\chi) \cup (V(B_{\chi'}) + \chi) \qquad \textit{for } \chi, \chi' \in \mathbf{Z}A. \tag{4.22}$$

2.

$$V(B_{\chi+\chi'}) = V(B_\chi) \cup (V(B_{\chi'}) + \chi) \qquad \textit{for } \chi, \chi' \in \mathbf{N}A. \tag{4.23}$$

Proof.

1. Let $p_\chi \in B_\chi$, $p_{\chi'} \in B_{\chi'}$, and $P_\chi \in S_\chi$, $P_{\chi'} \in S_{\chi'}$ be in the correspondence in Proposition 4.1. Then

$$\begin{aligned} P_\chi P_{\chi'} \partial^{\chi'_+} \partial^{\chi_+} &= P_\chi p_{\chi'}(s) \partial^{\chi'_-} \partial^{\chi_+} \\ &= p_{\chi'}(s-\chi) P_\chi \partial^{\chi_+} \partial^{\chi'_-} \\ &= p_{\chi'}(s-\chi) p_\chi(s) \partial^{\chi_-} \partial^{\chi'_-}. \end{aligned} \tag{4.24}$$

   Hence $p_{\chi'}(s-\chi)p_\chi(s) \in B_{\chi+\chi'}$.

2. Let $p_{\chi+\chi'} \in B_{\chi+\chi'}$ and $P_{\chi+\chi'} \in S_{\chi+\chi'}$ be in the correspondence in Proposition 4.1. Then

$$p_{\chi+\chi'} = P_{\chi+\chi'} \partial^{\chi'} \cdot \partial^\chi.$$

   Hence $p_{\chi+\chi'}(s) \in B_\chi$.

   Furthermore

$$p_{\chi+\chi'}(s+\chi)\partial^\chi = \partial^\chi p_{\chi+\chi'}(s) = \partial^\chi P_{\chi+\chi'} \partial^{\chi'} \partial^\chi.$$

   Hence $p_{\chi+\chi'}(s+\chi) = \partial^\chi P_{\chi+\chi'} \partial^{\chi'}$, which impies $p_{\chi+\chi'}(s+\chi) \in B_{\chi'}$.

▯

Proposition 4.7 *Let $\chi \in \mathbf{Z}A$. Let $p_\chi \in B_\chi$, $p_{-\chi} \in B_{-\chi}$, and $P_\chi \in S_\chi$, $P_{-\chi} \in S_{-\chi}$ be in the correspondence in Proposition 4.1. Then*

$$P_{-\chi} P_\chi = p_\chi(s+\chi) p_{-\chi}(s). \tag{4.25}$$

Proof.

$$\begin{aligned} P_{-\chi} P_\chi \partial^{\chi_+} &= P_{-\chi} p_\chi(s) \partial^{\chi_-} \\ &= p_\chi(s+\chi) P_{-\chi} \partial^{\chi_-} \\ &= p_\chi(s+\chi) p_{-\chi}(s) \partial^{\chi_+}. \end{aligned} \tag{4.26}$$

Divide it by ∂^{χ_+} to obtain the conclusion. ▯

For $\chi \in \mathbf{Z}A$, define an ideal $B_{-\chi,\chi}$ by

$$B_{-\chi,\chi} := \langle\, p_\chi(s+\chi)p_{-\chi}(s) \,|\, p_\chi \in B_\chi, p_{-\chi} \in B_{-\chi} \,\rangle. \tag{4.27}$$

Then the following proposition is immediate from the definition of $B_{-\chi,\chi}$.

Proposition 4.8 *1.*

$$V(B_{-\chi,\chi}) = (V(B_\chi) - \chi) \cup V(B_{-\chi}). \tag{4.28}$$

2.

$$V(B_{-\chi,\chi}) = V(B_{\chi,-\chi}) - \chi. \tag{4.29}$$

Theorem 4.9 *Let $\chi \in \mathbf{Z}A$. If $\beta \notin V(B_{-\chi,\chi})$, then two A-hypergeometric systems $M_A(\beta)$ and $M_A(\beta+\chi)$ are isomorphic.*

Proof. First note that $\beta \notin V(B_{-\chi,\chi})$ is equivalent to $\beta+\chi \notin V(B_{\chi,-\chi})$ by Proposition 4.8. Take polynomials $p_\chi \in B_\chi$ and $p_{-\chi} \in B_{-\chi}$ such that $p_\chi(\beta+\chi)p_{-\chi}(\beta) \neq 0$. Let $P_\chi \in S_\chi$, $P_{-\chi} \in S_{-\chi}$ be in the correspondence in Proposition 4.1. Then by Proposition 4.7, we have the following equalities:

$$\begin{aligned} P_{-\chi}P_\chi &= p_\chi(s+\chi)p_{-\chi}(s), & (4.30)\\ P_\chi P_{-\chi} &= p_{-\chi}(s-\chi)p_\chi(s). & (4.31) \end{aligned}$$

The multiplications by $P_{-\chi}, P_\chi$ respectively induce homomorphisms:

$$\begin{aligned} f &: M_A(\beta) \longrightarrow M_A(\beta+\chi), & (4.32)\\ g &: M_A(\beta+\chi) \longrightarrow M_A(\beta). & (4.33) \end{aligned}$$

Then

$$g \circ f = p_\chi(\beta+\chi)p_{-\chi}(\beta) id_{M_A(\beta)} \tag{4.34}$$

and

$$\begin{aligned} f \circ g &= p_{-\chi}((\beta+\chi)-\chi)p_\chi(\beta+\chi) id_{M_A(\beta+\chi)} & (4.35)\\ &= p_{-\chi}(\beta)p_\chi(\beta+\chi) id_{M_A(\beta+\chi)}. & (4.36) \end{aligned}$$

Hence f and g are isomorphisms. $\square$

Now we are ready to prove the if-part of our main theorem.

Proof of the if-part of Theorem 2.1.

We suppose that $E_\tau(\beta) = E_\tau(\beta')$ for all faces. Let $\chi := \beta' - \beta$. We claim $\beta \notin V(B_{-\chi})$. Assume the contrary. Then by Theorem 4.5, there exists a standard pair $(u,\tau) \in \mathcal{S}(M_{-\chi})$ such that $\beta - Au \in \mathbf{k}(A \cap \tau)$. The equality $E_\tau(\beta) = E_\tau(\beta')$ implies that there exists $v \in \mathbf{N}^n$ such that $\beta - \beta' = A(u-v)$. Hence the intersection of $Au + \mathbf{N}(A \cap \tau)$ with $(\beta - \beta') + \mathbf{N}A$ is not empty. This contradicts the standardness of (u,τ). We have thus proved $\beta \notin V(B_{-\chi})$. By symmetry we have $\beta' \notin V(B_\chi)$, which is equivalent to $\beta \notin V(B_\chi) - \chi$. Hence $\beta \notin V(B_{-\chi,\chi})$ by Proposition 4.8. From Theorem 4.9 we conclude $M_A(\beta)$ is isomorphic to $M_A(\beta')$. $\square$

As a corollary of the proof of the if-part of Theorem 2.1, we obtain the following.

Corollary 4.10 *If two A-hypergeometric systems $M_A(\beta)$ and $M_A(\beta')$ are isomorphic, then there exists an operator $P \in S_{\beta'-\beta}$ such that the multiplication by P from the right induces an isomorphism from $M_A(\beta)$ to $M_A(\beta')$.*

5 Normal case

In this section, we consider the normal case:

$$\mathbf{N}A = \mathbf{Z}A \cap \mathbf{Q}_{\geq 0}A. \tag{5.37}$$

Many important examples are known to be normal, such as Aomoto-Gel'fand systems, the A-hypergeometric systems corresponding to ${}_{p+1}F_p$, Lauricella functions, etc. (see [9], [10]). It will turn out below that the parameter space can be classified in terms of the primitive integral support functions F_σ in the normal case.

Lemma 5.1 *Assume A to be normal. Then we have the following.*

1. *$(\mathbf{Q}(A \cap \tau)) \cap \mathbf{Z}A$ equals $\mathbf{Z}(A \cap \tau)$ for all faces τ.*
2. *$F_\sigma(\mathbf{N}A) = \mathbf{N}$ for all facets σ.*
3. *For a face τ,*

$$\mathbf{N}A + \mathbf{Z}(A \cap \tau) = \mathbf{Z}A \cap \bigcap_{\sigma\text{: facet} \supset \tau} (\mathbf{N}A + \mathbf{k}(A \cap \sigma)). \tag{5.38}$$

Proof. (1) Let $\chi \in (\mathbf{Q}(A \cap \tau)) \cap \mathbf{Z}A$. Add a vector $\chi' \in \mathbf{N}(A \cap \tau)$ to χ so that $\chi + \chi' \in \mathbf{Q}_{\geq 0}(A \cap \tau)$. By the normality, we see $\chi + \chi' \in \mathbf{N}(A \cap \tau)$. Hence χ belongs to $\mathbf{Z}(A \cap \tau)$.

(2) Let $\chi \in \mathbf{Z}A$ satisfy $F_\sigma(\chi) = 1$. For $\sigma' \neq \sigma$, there exists $a_j \in \sigma \setminus \sigma'$. Hence there exists $\chi' \in \mathbf{N}(A \cap \sigma)$ such that $F_{\sigma'}(\chi + \chi') \geq 0$ for all facets σ'. By the normality, $\chi + \chi' \in \mathbf{N}A$. Since $F_\sigma(\chi + \chi') = 1$, we obtain $F_\sigma(\mathbf{N}A) = \mathbf{N}$.

(3) Let $\chi \in \mathbf{Z}A$ satisfy $F_\sigma(\chi) \geq 0$ for all facets containing τ. For a facet σ not containing the face τ, there exists $a_j \in \tau \setminus \sigma$. Hence there exists a vector $\chi' \in \mathbf{N}(A \cap \tau)$ such that $F_\sigma(\chi + \chi') \geq 0$ for all facets σ of the cone $\mathbf{Q}_{\geq 0}A$. By the normality, $\chi + \chi' \in \mathbf{N}A$, and thus $\chi \in \mathbf{N}(A \setminus A \cap \tau) + \mathbf{Z}(A \cap \tau)$. $\Box$

Theorem 5.2 *Let $\beta, \beta' \in \mathbf{k}^d$. Then $M_A(\beta) \simeq M_A(\beta')$ if and only if $\beta - \beta' \in \mathbf{Z}A$ and $\{\sigma : \text{facet}, F_\sigma(\beta) \in \mathbf{N}\} = \{\sigma : \text{facet}, F_\sigma(\beta') \in \mathbf{N}\}$.*

Proof. By Proposition 2.2 (3), the only-if-part follows from Theorem 2.1.

Next we prove the if-part. Suppose $\beta - \beta' \in \mathbf{Z}A$ and $\{\sigma : \text{facet}, F_\sigma(\beta) \in \mathbf{N}\} = \{\sigma : \text{facet}, F_\sigma(\beta') \in \mathbf{N}\}$. By Lemma 5.1 (1), (2), and Propositions 2.2, 2.3, we obtain $E_\sigma(\beta) = E_\sigma(\beta')$ for all facets. By Lemma 5.1 (3), the if-part follows from Theorem 2.1. $\Box$

Example 5.3 *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Let $\beta \in \mathbf{Z}A = \mathbf{Z}^d$. Then by Theorem 5.2, the A-hypergeometric system $M_A(\beta)$ is isomorphic to

$$\begin{array}{rll}
M_A({}^t(0,0,0)) & \textit{if} & \beta_1 \geq 0, \beta_2 \geq 0, \beta_1+\beta_3 \geq 0, \beta_2+\beta_3 \geq 0, \\
M_A({}^t(-1,0,1)) & \textit{if} & \beta_1 < 0, \beta_2 \geq 0, \beta_1+\beta_3 \geq 0, \beta_2+\beta_3 \geq 0, \\
M_A({}^t(0,-1,1)) & \textit{if} & \beta_1 \geq 0, \beta_2 < 0, \beta_1+\beta_3 \geq 0, \beta_2+\beta_3 \geq 0, \\
M_A({}^t(0,1,-1)) & \textit{if} & \beta_1 \geq 0, \beta_2 \geq 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 \geq 0, \\
M_A({}^t(1,0,-1)) & \textit{if} & \beta_1 \geq 0, \beta_2 \geq 0, \beta_1+\beta_3 \geq 0, \beta_2+\beta_3 < 0, \\
M_A({}^t(-1,-1,1)) & \textit{if} & \beta_1 < 0, \beta_2 < 0, \beta_1+\beta_3 \geq 0, \beta_2+\beta_3 \geq 0, \\
M_A({}^t(-1,0,0)) & \textit{if} & \beta_1 < 0, \beta_2 \geq 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 \geq 0, \\
M_A({}^t(0,-1,0)) & \textit{if} & \beta_1 \geq 0, \beta_2 < 0, \beta_1+\beta_3 \geq 0, \beta_2+\beta_3 < 0, \\
M_A({}^t(0,0,-1)) & \textit{if} & \beta_1 \geq 0, \beta_2 \geq 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 < 0, \\
M_A({}^t(-2,-1,1)) & \textit{if} & \beta_1 < 0, \beta_2 < 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 \geq 0, \\
M_A({}^t(-1,-2,1)) & \textit{if} & \beta_1 < 0, \beta_2 < 0, \beta_1+\beta_3 \geq 0, \beta_2+\beta_3 < 0, \\
M_A({}^t(-1,0,-1)) & \textit{if} & \beta_1 < 0, \beta_2 \geq 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 < 0, \\
M_A({}^t(0,-1,-1)) & \textit{if} & \beta_1 \geq 0, \beta_2 < 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 < 0, \\
M_A({}^t(-1,-1,0)) & \textit{if} & \beta_1 < 0, \beta_2 < 0, \beta_1+\beta_3 < 0, \beta_2+\beta_3 < 0.
\end{array}$$

6 Monomial curve case

In this section, we conider $d = 2$ case. Let

$$A \quad = \quad \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & i_2 & i_3 & \cdots & i_{n-1} & i_n \end{pmatrix}$$

with $0 < i_2 < i_3 < \cdots < i_n$ relative prime integers. Put $F_{\sigma_1}(s) = s_2$ and $F_{\sigma_2}(s) = i_n s_1 - s_2$.

We denote by $\mathcal{E}(A)$ the set of holes, i.e.,

$$\begin{aligned}
\mathcal{E}(A) : \;&=\; ((\mathbf{N}A + \mathbf{Z}a_1) \cap (\mathbf{N}A + \mathbf{Z}a_n)) \setminus \mathbf{N}A &(6.39)\\
&=\; \{\, \beta \mid E_{\mathbf{Q}_{\geq 0}A}(\beta) = \{0\},\, E_{\sigma_1}(\beta) = \{0\}, \\
&\qquad\qquad E_{\sigma_2}(\beta) = \{0\},\, E_{\{0\}}(\beta) = \emptyset \,\}. &(6.40)
\end{aligned}$$

The rank of $M_A(\beta)$ is d or $d+1$, and it equals $d+1$ if and only if $\beta \in \mathcal{E}(A)$ (see [2], [13]).

Lemma 6.1 *For any face τ,*

$$\mathbf{Z}A \cap (\mathbf{k}(A \cap \tau)) = \mathbf{Z}(A \cap \tau). \tag{6.41}$$

Proof. When τ is the whole cone $\mathbf{Q}_{\geq 0}A$ or the origin $\{0\}$, the statement is trivial.

Note that β belongs to $\mathbf{Z}A$ if and only if $F_{\sigma_1}(\beta) \in \mathbf{Z}$, $F_{\sigma_2}(\beta) \in \mathbf{Z}$, and $F_{\sigma_1}(\beta) + F_{\sigma_2}(\beta) \in i_n\mathbf{Z}$. Suppose $\beta \in \mathbf{Z}A \cap (\mathbf{k}(A \cap \sigma_1))$. Then $F_{\sigma_2}(\beta) \in i_n\mathbf{Z}$. When $F_{\sigma_2}(\beta) = di_n$, we have $\beta = da_n$. ▯

Corollary 6.2

$$\mathcal{E}(A) = \{\, \beta \in \mathbf{Z}A \,|\, F_{\sigma_1}(\beta) \in F_{\sigma_1}(\mathbf{N}A),\ F_{\sigma_2}(\beta) \in F_{\sigma_2}(\mathbf{N}A) \,\} \setminus \mathbf{N}A. \tag{6.42}$$

Proof. This is immediate from Lemma 6.1. ▯

Theorem 2.1 in the monomial curve case is as follows.

Theorem 6.3 *Let $\beta, \beta' \in \mathbf{k}^d$.*

1. *Suppose $\beta \notin \mathcal{E}(A)$. Then $M_A(\beta')$ is isomorphic to $M_A(\beta)$ if and only if $\beta - \beta' \in \mathbf{Z}A$, $\beta' \notin \mathcal{E}(A)$, and $\{\sigma_i : F_{\sigma_i}(\beta) \in F_{\sigma_i}(\mathbf{N}A)\} = \{\sigma_i : F_{\sigma_i}(\beta') \in F_{\sigma_i}(\mathbf{N}A)\}$.*

2. *Suppose $\beta \in \mathcal{E}(A)$. Then $M_A(\beta')$ is isomorphic to $M_A(\beta)$ if and only if $\beta \in \mathcal{E}(A)$.*

Proof. (2) directly follows from Theorem 2.1.

The only-if-part of (1) follows from Theorem 2.1 by Proposition 2.2 (3). Next suppose that $\beta - \beta' \in \mathbf{Z}A$, $\beta, \beta' \notin \mathcal{E}(A)$, and that $\{\sigma_i : F_{\sigma_i}(\beta) \in F_{\sigma_i}(\mathbf{N}A)\} = \{\sigma_i : F_{\sigma_i}(\beta') \in F_{\sigma_i}(\mathbf{N}A)\}$. Then by Lemma 6.1, Proposition 2.2 (3), and Proposition 2.3 (2), we have $E_{\sigma_i}(\beta) = E_{\sigma_i}(\beta')$ for $i = 1, 2$. Moreover we know $E_{\{0\}}(\beta), E_{\{0\}}(\beta') = \emptyset$ from Proposition 2.2 (2). Hence $M_A(\beta)$ and $M_A(\beta')$ are isomorphic by Theorem 2.1. ▯

Example 6.4 *([13, Chapter 4]) Let*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 7 & 9 \end{pmatrix}.$$

Then

$$F_{\sigma_1}(\mathbf{N}A) = \{\, 0, 2, 4, 6, 7, 8, 9, \ldots \},\tag{6.43}$$

and

$$F_{\sigma_2}(\mathbf{N}A) = \{\, 0, 2, 4, 5, 6, 7, 8, 9, \ldots \},\tag{6.44}$$

Parameters in $\mathbf{Z}A = \mathbf{Z}^2$ are decomposed into five parts according to the isomorphism classes of their corresponding A-hypergeometric systems:

1. $\mathbf{N}A$,

2. $\{\, {}^t(\beta_1, \beta_2) \,|\, \beta_2 \in F_{\sigma_1}(\mathbf{N}A),\ 9\beta_1 - \beta_2 \notin F_{\sigma_2}(\mathbf{N}A) \,\}$,

3. $\{\, {}^t(\beta_1, \beta_2) \,|\, \beta_2 \notin F_{\sigma_1}(\mathbf{N}A),\ 9\beta_1 - \beta_2 \in F_{\sigma_2}(\mathbf{N}A) \,\}$,

4. $\{\, {}^t(\beta_1, \beta_2) \,|\, \beta_2 \notin F_{\sigma_1}(\mathbf{N}A),\ 9\beta_1 - \beta_2 \notin F_{\sigma_2}(\mathbf{N}A) \,\}$,

5. $\mathcal{E}(A) = \{{}^t(2, 10), {}^t(2, 12), {}^t(3, 19)\}$ *: the set of holes.*

7 Final remark

Thanks to Theorem 2.1, all D-invariants of A-hypergeometric systems can be described in terms of $E_\tau(\beta)$; the characteristic cycles (in particular, the rank), the monodromy representations, etc. One of most recent results is given by Tsushima ([15]) on Laurent polynomial solutions. He has proved that the vector space of Laurent polynomial solutions of $M_A(\beta)$ has a basis consisting of canonical series whose negative support corresponds to a face τ of $\mathbf{Q}_{\geq 0}A$ such that $\dim \tau = |\{ a_j \mid a_j \in \tau \}|$, and that $0 \in E_\tau(\beta)$ but $0 \notin E_{\tau'}(\beta)$ for any $\tau' \subset \tau$. In particular, the dimension of the vector space of Laurent polynomial solutions equals the cardinality of the set of such faces. This is a generalization of the corresponding result by Cattani, D'Andrea and Dickenstein ([2]) in the monomial curve case.

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